

ADMISSIBLE PICTURES AND $U_q(\mathfrak{gl}(m, n))$ -LITTLEWOOD-RICHARDSON TABLEAUX

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ABSTRACT. We construct a natural bijection between the set of admissible pictures and the set of $U_q(\mathfrak{gl}(m, n))$ -Littlewood-Richardson tableaux.

INTRODUCTION

The notion of *pictures* was introduced by James and Peel [5] and Zelevinsky [9]. In [8], Nakashima and Shimozono considered the notion of *admissible pictures* and showed that there exists a natural bijection between the set of admissible pictures and the set of $U_q(\mathfrak{gl}(r))$ -Littlewood-Richardson tableaux. More precisely, let Y, W, Z be Young diagrams with at most r rows such that $|Y| + |W| = |Z|$, and let A, A' be admissible orders on $Z/Y, W$, respectively. Then Nakashima and Shimozono constructed an explicit natural bijection between the set $\mathbf{P}(W, Z/Y; A, A')$ of (A, A') -admissible pictures and the set $\mathbf{B}(W)_Y^Z[A']$ of $U_q(\mathfrak{gl}(r))$ -Littlewood-Richardson tableaux. This result was already obtained in [2] and [3] by a purely combinatorial method. Nakashima and Shimozono gave an alternative proof using the theory of $U_q(\mathfrak{gl}(r))$ -crystals.

In this paper, we generalize the main result of [8] to the case when W is a skew Young diagram (Theorem 2.7). Our proof follows the outline given in [8] with some necessary modifications to deal with semistandard skew tableaux.

Moreover, we introduce the notion of $U_q(\mathfrak{gl}(m, n))$ -*Littlewood-Richardson tableaux* arising from the theory of $U_q(\mathfrak{gl}(m, n))$ -crystals, and show that there exists a natural bijection between the set of admissible pictures and the set of $U_q(\mathfrak{gl}(m, n))$ -Littlewood-Richardson tableaux (Theorem 3.3). Namely, let Y, W, Z be (m, n) -hook Young diagrams such that $|Y| + |W| = |Z|$ and let A, A' be admissible orders on $W, Z/Y$, respectively. Denote by $LR(Y, W)^Z[A']$ the set of $U_q(\mathfrak{gl}(m, n))$ -Littlewood-Richardson tableaux associated with (Y, W, Z) and A' . We construct an explicit natural bijection between $\mathbf{P}(Z/Y, W; A, A')$ and

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$LR(Y, W)^Z[A']$. As a corollary, we show that the $U_q(\mathfrak{gl}(m, n))$ -Littlewood-Richardson coefficients and the $U_q(\mathfrak{gl}(r))$ -Littlewood-Richardson coefficients are the same.

1. ADMISSIBLE PICTURES

We first recall some basic notions that are used in this paper. A *Young diagram* is a collection of boxes arranged in left-justified rows with a weakly decreasing number of boxes in each row as we go down. A Young diagram may be identified with a *partition* $Y = (Y_1 \geq Y_2 \geq \cdots)$, where Y_i is the number of boxes in the i -th row. For Young diagrams Y and Z with $Z \supset Y$, we denote by Z/Y the *skew Young diagram* obtained by removing Y from Z . For a skew Young diagram Y , the *size* of Y , denoted by $|Y|$, is defined to be the total number of boxes in Y . A *Young tableau* (resp. *skew tableau*) T is a filling of a Young diagram (resp. skew Young diagram) with positive integers. We say that T is *semistandard* if

- (i) the entries in each row are weakly increasing from left to right,
- (ii) the entries in each column are strictly increasing from top to bottom.

The (skew) Young diagram Y is called the *shape* of T . We often write $\text{sh}(T) = Y$.

The notion of *pictures* was first introduced by James and Peel [5] and Zelevinsky [9]. In this paper, we use a generalized definition given by Nakashima and Shimozono [8].

Definition 1.1. Let X and Y be subsets of $\mathbb{N} \times \mathbb{N}$, where \mathbb{N} is the set of positive integers.

- (a) For $(a, b), (c, d) \in X$, we define

$$(a, b) \leq_P (c, d) \text{ if and only if } a \leq c, b \leq d.$$

- (b) A total order \leq_A on X is said to be *admissible* if

$$(a, b) \leq_A (c, d) \text{ whenever } a \leq c, b \geq d.$$

Thus $(a, b) \leq_A (c, d)$ whenever (a, b) lies in the northeast of (c, d) .

Example 1.2. In this example, we introduce two typical examples of admissible order.

- (a) The *Middle Eastern order* \leq_{ME} on X is defined by

$$(a, b) \leq_{ME} (c, d) \text{ if and only if } a < c, \text{ or } a = c, b \geq d.$$

- (b) The *Far Eastern order* \leq_{FE} on X is defined by

$$(a, b) \leq_{FE} (c, d) \text{ if and only if } b > d, \text{ or } b = d, a \leq c.$$

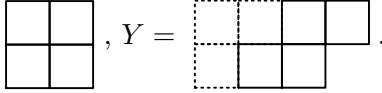
Note that a (skew) Young diagram Y may be regarded as a subset of $\mathbb{N} \times \mathbb{N}$ by identifying the box in the i -th row and j -th column with $(i, j) \in \mathbb{N} \times \mathbb{N}$. Hence we may consider the notion of admissible orders on Y .

Definition 1.3. Let X, Y be subsets of $\mathbb{N} \times \mathbb{N}$ and let \leq_A (resp. $\leq_{A'}$) be an admissible order on Y (resp. on X).

- (a) A map $f : X \rightarrow Y$ is *PA-standard* if $f(a, b) \leq_A f(c, d)$ whenever $(a, b) \leq_P (c, d)$.
- (b) A bijection $f : X \rightarrow Y$ is called an (A, A') -*admissible picture* if $f : X \rightarrow Y$ is *PA-standard* and $f^{-1} : Y \rightarrow X$ is *PA'-standard*.

We denote by $\mathbf{P}(X, Y; A, A')$ the set of all (A, A') -admissible pictures from X to Y . Since (skew) Young diagrams may be considered as subsets of $\mathbb{N} \times \mathbb{N}$, for any pair of (skew) Young diagrams Y and W with $|Y| = |W|$, we may define the notion of admissible pictures from Y to W and vice versa.

Example 1.4. Let $X = \{(1, 1), (1, 2), (2, 1), (2, 2)\}$ and $Y = \{(1, 3), (1, 4), (2, 2), (2, 3)\}$ be skew Young diagrams. Pictorially, we have $X =$



Define a map $f : X \rightarrow Y$ by

$$f(1, 1) = (1, 4), \quad f(1, 2) = (1, 3), \quad f(2, 1) = (2, 3), \quad f(2, 2) = (2, 2).$$

Then it is easy to verify that f is an (A, A') -admissible picture for any admissible orders A and A' .

Let $f : X \rightarrow Y$ be an (A, A') -admissible picture. We denote by f_1 and f_2 the 1st and 2nd coordinate functions, respectively. That is, if $f(i, j) = (a, b)$, then $f_1(i, j) = a$, $f_2(i, j) = b$. For simplicity, we often write $f = (f_1, f_2)$.

2. $U_q(\mathfrak{gl}(r))$ -LITTLEWOOD-RICHARDSON TABLEAUX

In this section, we review the main result of [8] on the 1-1 correspondence between the set of admissible pictures and the set of $U_q(\mathfrak{gl}(r))$ -Littlewood-Richardson tableaux. More details on $U_q(\mathfrak{gl}(r))$ -crystals can be found in [4].

Let

$$\mathbf{B} : \boxed{1} \xrightarrow{1} \boxed{2} \xrightarrow{2} \cdots \xrightarrow{n-2} \boxed{n-1} \xrightarrow{n-1} \boxed{n}$$

be the crystal of the vector representation of $U_q(\mathfrak{gl}(r))$. For a (skew) Young diagram Y , we denote by $\mathbf{B}(Y)$ the set of all semistandard tableaux of shape Y with entries in $\{1, 2, \dots, r\}$. If $|Y| = N$, a semistandard tableau $T \in \mathbf{B}(Y)$ can be identified with the element

$$\boxed{b_1} \otimes \boxed{b_2} \otimes \cdots \otimes \boxed{b_N} \in \mathbf{B}^{\otimes N},$$

where b_1, b_2, \dots, b_N are the entries of T listed by a given admissible order A . Thus we get an embedding

$$R_A : \mathbf{B}(Y) \rightarrow \mathbf{B}^{\otimes N} \quad \text{given by} \quad T \mapsto \boxed{b_1} \otimes \boxed{b_2} \otimes \cdots \otimes \boxed{b_N},$$

which is called the *admissible reading of $\mathbf{B}(Y)$ by A* .

For example, the *Middle-Eastern reading* $R_{ME}(T)$ of $T \in \mathbf{B}(Y)$ reads the entries by moving across the rows from right to left and top to bottom. On the other hand, the *Far-Eastern reading* $R_{FE}(T)$ of $T \in \mathbf{B}(Y)$ proceeds down the columns from top to bottom and from right to left.

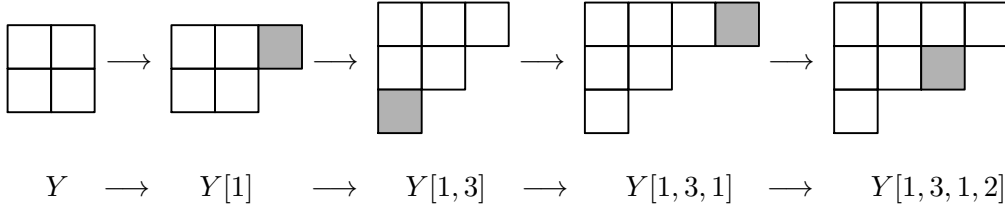
Example 2.1.

$$\begin{aligned} R_{ME} \left(\begin{array}{|c|c|c|} \hline 2 & 2 & 5 \\ \hline 3 & 4 & \\ \hline \end{array} \right) &= \boxed{5} \otimes \boxed{2} \otimes \boxed{2} \otimes \boxed{4} \otimes \boxed{3} \\ R_{FE} \left(\begin{array}{|c|c|c|} \hline 2 & 2 & 5 \\ \hline 3 & 4 & \\ \hline \end{array} \right) &= \boxed{5} \otimes \boxed{2} \otimes \boxed{4} \otimes \boxed{2} \otimes \boxed{3}. \end{aligned}$$

An admissible reading R_A provides $\mathbf{B}(Y)$ with a $U_q(\mathfrak{gl}(r))$ -crystal structure by the tensor product rule. In [4], it was shown that the $U_q(\mathfrak{gl}(r))$ -crystal structure on $\mathbf{B}(Y)$ does not depend on the choice of admissible reading R_A .

Let Y be a Young diagram. We denote by $Y[j]$ the diagram obtained from Y by adding a box at the j -th row. If $Y[j]$ is a Young diagram, then $\mathbf{B}(Y[j])$ is the set of semistandard tableaux of shape $Y[j]$. If $Y[j]$ is *not* a Young diagram, we define $\mathbf{B}(Y[j]) = \emptyset$. More generally, $Y[j_1, \dots, j_N]$ is the diagram obtained from $Y[j_1, \dots, j_{N-1}]$ by adding a box at the j_N -th row and $\mathbf{B}(Y[j_1, \dots, j_N])$ is the set of semistandard tableaux of shape $Y[j_1, \dots, j_N]$ if $Y[j_1, \dots, j_k]$ is a Young diagram for all $k = 1, \dots, N$. We define $\mathbf{B}(Y[j_1, \dots, j_N]) = \emptyset$ if $Y[j_1, \dots, j_k]$ is not a Young diagram for some $k = 1, \dots, N$.

Example 2.2. For $Y = \begin{array}{|c|c|} \hline & \\ \hline & \\ \hline \end{array}$, $Y[1, 3, 1, 2]$ is obtained as follows:



With this notation, Nakashima obtained the following decomposition of the tensor product of $U_q(\mathfrak{gl}(r))$ -crystals.

Proposition 2.3. [7] *Let Y and W be Young diagrams with at most r rows. Then there exists a $U_q(\mathfrak{gl}(r))$ -crystal isomorphism*

$$(2.1) \quad \mathbf{B}(Y) \otimes \mathbf{B}(W) \cong \bigoplus_{\substack{T \in \mathbf{B}(W), \\ R_{FE}(T) = \boxed{j_1} \otimes \cdots \otimes \boxed{j_N}}} \mathbf{B}(Y[j_1, \dots, j_N]).$$

By the same argument in [6, Proposition 4.13], one can show that $Y[j_1, \dots, j_N]$ are the same for all admissible reading $R_A(T) = \boxed{j_1} \otimes \cdots \otimes \boxed{j_N}$. Hence the tensor product decomposition (2.1) can be restated as

$$(2.2) \quad \mathbf{B}(Y) \otimes \mathbf{B}(W) \cong \bigoplus_{\substack{T \in \mathbf{B}(W), \\ R_A(T) = \boxed{j_1} \otimes \cdots \otimes \boxed{j_N}}} \mathbf{B}(Y[j_1, \dots, j_N]),$$

where R_A is an arbitrary admissible reading.

Motivated by this, we make the following definitions. Let Y, W, Z be the Young diagrams with at most r rows such that $|Y| + |W| = |Z|$, and let A be an admissible order on W . We define $\mathbf{B}(W)_Y^Z[A]$ to be the set of semistandard tableaux of shape W satisfying the following conditions:

$$(2.3) \quad \begin{aligned} &\text{if } R_A(T) = \boxed{j_1} \otimes \cdots \otimes \boxed{j_N}, \text{ then} \\ &\text{(i) } Y[j_1, \dots, j_k] \text{ is a Young diagram for all } k = 1, \dots, N, \\ &\text{(ii) } Y[j_1, \dots, j_N] = Z. \end{aligned}$$

Since the decomposition (2.2) is unique, the set $\mathbf{B}(W)_Y^Z[A]$ are the same for all admissible orders A on W . Hence we may write

$$\mathbf{B}(W)_Y^Z = \mathbf{B}(W)_Y^Z[A]$$

for any admissible order A on W .

Definition 2.4.

(a) A semistandard tableau T in $\mathbf{B}(W)_Y^Z$ is called a $U_q(\mathfrak{gl}(r))$ -Littlewood-Richardson tableau associated with the triple (Y, W, Z) .

(b) The number $c_{Y,W}^Z = |\mathbf{B}(W)_Y^Z|$ is called the $U_q(\mathfrak{gl}(r))$ -Littlewood-Richardson coefficient associated with (Y, W, Z) .

Now we explain the main result of [8]. Let Y be a Young diagram and let T be a semistandard tableau in $\mathbf{B}(Y)$. We denote by T_{ij} the (i, j) -entry of T . For each $k \in \mathbb{N}$, we define

$$T^{(k)} = \{(i, j) \in Y \mid T_{ij} = k\}.$$

Since no entry can occur more than once in any column of T , we may write

$$T^{(k)} = \{(a_1, b_1), \dots, (a_r, b_r)\},$$

where $a_1 \leq \dots \leq a_r$, $b_1 > \dots > b_r$. Define

$$p(T; a_i, b_i) = i \quad \text{for } (a_i, b_i) \in T^{(k)}.$$

Hence if T is a semistandard tableau in $\mathbf{B}(Y)$ with $T_{ij} = k$, then $p(T; i, j)$ is equal to the number of boxes with entry k that lie in the right of T_{ij} (including T_{ij} itself).

Example 2.5. For $T = \begin{array}{|c|c|c|} \hline 1 & 1 & 2 \\ \hline 2 & 2 & \\ \hline \end{array}$, we have

$$T^{(1)} = \{(1, 2), (1, 1)\}, \quad T^{(2)} = \{(1, 3), (2, 2), (2, 1)\},$$

and

$$p(T; 1, 1) = 2, \quad p(T; 1, 2) = 1, \quad p(T; 1, 3) = 1, \quad p(T; 2, 1) = 3, \quad p(T; 2, 2) = 2.$$

Proposition 2.6. [8] *Let Y, W, Z be Young diagrams with at most r rows such that $|Y| + |W| = |Z|$, and let A, A' be admissible orders on $Z/Y, W$, respectively. Then there exists a natural bijection*

$$\Phi : \mathbf{P}(W, Z/Y; A, A') \longrightarrow \mathbf{B}(W)_Y^Z[A']$$

defined by

$$(2.4) \quad \Phi(f)_{ij} = f_1(i, j) \quad \text{for } f = (f_1, f_2) \in \mathbf{P}(W, Z/Y; A, A').$$

The inverse map

$$\Psi : \mathbf{B}(W)_Y^Z[A'] \longrightarrow \mathbf{P}(W, Z/Y; A, A')$$

is given by

$$(2.5) \quad \Psi(T)(i, j) = (T_{ij}, Y_{T_{ij}} + p(T; i, j)) \quad \text{for } T \in \mathbf{B}(W)_Y^Z[A'].$$

Let Y, Z be Young diagrams with at most r rows and let W be a skew Young diagram such that $|Y| + |W| = |Z|$. We define $\mathbf{B}(W)_Y^Z[A]$ to be the set of all semistandard skew tableaux T in $\mathbf{B}(W)$ satisfying the condition (2.3). For a semistandard skew tableau T of shape W , we define $p(T; i, j)$ to be the number of boxes with entry T_{ij} lying in the right of the (i, j) -position (including the box at the (i, j) -position). Our first main result is the following generalization of Proposition 2.6.

Theorem 2.7. *Let Y, Z be Young diagrams with at most r rows and let W be a skew Young diagram such that $|Y| + |W| = |Z|$. Let A and A' be arbitrary admissible orders on Z/Y and W , respectively. Then there exist natural bijections*

$$\begin{aligned}\Phi : \mathbf{P}(W, Z/Y; A, A') &\longrightarrow \mathbf{B}(W)_Y^Z[A'], \\ \Psi : \mathbf{B}(W)_Y^Z[A'] &\longrightarrow \mathbf{P}(W, Z/Y; A, A')\end{aligned}$$

defined by (2.4) and (2.5), which are the inverses to each other.

Proof. Our proof follows the outline given in [8]. The key ingredient of our generalization is the following almost self-obvious lemma on skew Young diagrams.

Lemma 2.8. *Let W be a skew Young diagram. If $(a, b), (c, d) \in W$ and $a \leq c, b \leq d$, then every (x, y) satisfying*

$$(a, b) \leq_P (x, y) \leq_P (c, d)$$

lies in W .

We now proceed to prove our theorem in 3 steps.

Step 1: *The map Φ is well-defined.*

Let $f = (f_1, f_2) \in \mathbf{P}(W, Z/Y; A, A')$. We first show that $\Phi(f)$ is a semistandard skew tableau of shape W . That is, we show

- (i) $f_1(i, j) < f_1(i+1, j)$ for all $(i, j) \in W$,
- (ii) $f_1(i, j) \leq f_1(i, j+1)$ for all $(i, j) \in W$.

The condition (i) can be verified by the same argument in [8, Proposition 5.1]. We will prove the condition (ii) using the induction on i . Let

$$i_0 = \min\{i \in \mathbb{N} \mid (i, j) \in W, (i, j+1) \in W\}.$$

Thus W has more than two boxes in the i_0 -th row for the first time.

If $i = i_0$, suppose on the contrary that $f_1(i_0, j) > f_1(i_0, j+1)$. Then by [8, Lemma 5.2], we have $f_2(i_0, j) > f_2(i_0, j+1)$. Moreover, by [8, Lemma 5.3], there exists a unique $(k, l) \in W$ satisfying

$$(2.6) \quad k < i_0, \quad l \leq j, \quad f_1(k, l) = f_1(i_0, j+1), \quad f_2(k, l) = f_2(i_0, j).$$

Since

$$(k, l) \leq_P (i_0 - 1, j) \leq_P (i_0, j) \quad \text{for } (k, l), (i_0, j) \in W,$$

by Lemma 2.8, we have $(i_0 - 1, j) \in W$. Applying Lemma 2.8 to $(i_0 - 1, j)$ and $(i_0, j+1)$, we get $(i_0 - 1, j+1) \in W$. Hence we have $(i_0 - 1, j) \in W$ and $(i_0 - 1, j+1) \in W$, which is a contradiction to the minimality of i_0 . Therefore, we conclude $f_1(i_0, j) \leq f_1(i_0, j+1)$ for all j with $(i_0, j), (i_0, j+1) \in W$.

If $i > i_0$, by a similar argument in the proof of [8, Proposition 5.1 (i)] using Lemma 2.8 whenever necessary, one can verify the condition (ii). Hence $\Phi(f)$ is a semistandard skew tableau of shape W .

Using almost the same argument in the proof of [8, Proposition 5.1 (ii)], it is straightforward to verify that $\Phi(f)$ satisfies the condition (2.3) with respect to A' . Therefore, Φ is well-defined.

Step 2: *The map Ψ is well-defined.*

Let $T \in \mathbf{B}(W)_Y^Z[A']$. One can easily verify that $\Psi(T)$ is a bijection from W to Z/Y . It remains to show $\Psi(T)$ is an (A, A') -admissible picture. For this purpose, one can verify that almost all the arguments in the proof of [8, Proposition 6.1] work in our case as well. The only difference is that, for a skew Young diagram W , the $U_q(\mathfrak{gl}(r))$ -crystal $\mathbf{B}(W)$ may not be connected. However, since the boxes in different connected components are not comparable with respect to P , it suffices to show that $\Psi(T)$ is PA -standard on each connected component, which can be checked in a straightforward manner combining Lemma 2.8 and the proof of [8, Proposition 6.1 (3)].

Step 3: *Φ and Ψ are inverses to each other.*

The same argument in [8, Section 7] works in our case as well. □

3. $U_q(\mathfrak{gl}(m, n))$ -LITTLEWOOD-RICHARDSON TABLEAUX

In this section, we will prove the main result of this paper. We will define the notion of $U_q(\mathfrak{gl}(m, n))$ -Littlewood-Richardson tableaux and show that there exists a natural bijection between the set of admissible pictures and the set of $U_q(\mathfrak{gl}(m, n))$ -Littlewood-Richardson tableaux. One may refer to [1] and [6] for more details on $U_q(\mathfrak{gl}(m, n))$ -crystals.

Let

$$\mathfrak{B} : \boxed{1} \xrightarrow{1} \boxed{2} \xrightarrow{2} \cdots \xrightarrow{m-1} \boxed{m} \xrightarrow{m} \boxed{\bar{1}} \xrightarrow{\bar{1}} \boxed{\bar{2}} \xrightarrow{\bar{2}} \cdots \xrightarrow{\bar{n-1}} \boxed{\bar{n}}$$

be the crystal of the vector representation of $U_q(\mathfrak{gl}(m, n))$. We define an ordering on \mathfrak{B} by

$$1 < 2 < \cdots < m < \bar{1} < \bar{2} < \cdots < \bar{n},$$

and set $\mathfrak{B}_+ = \{1, 2, \dots, m\}$, $\mathfrak{B}_- = \{\bar{1}, \bar{2}, \dots, \bar{n}\}$.

Definition 3.1. Let Y be a skew Young diagram. A $\mathfrak{gl}(m, n)$ -semistandard skew tableau of shape Y is a tableau T obtained from Y by filling the boxes with entries from \mathfrak{B} satisfying the following conditions:

- (i) the entries in each row and column are weakly increasing,
- (ii) the entries in \mathfrak{B}_+ are strictly increasing in each column,

(iii) the entries in \mathfrak{B}_- are strictly increasing in each row.

We denote by $\mathfrak{B}(Y)$ the set of all $\mathfrak{gl}(m, n)$ -semistandard tableaux of shape Y . In [1], it was shown that an admissible reading provides $\mathfrak{B}(Y)$ with a $U_q(\mathfrak{gl}(m, n))$ -crystal structure and it does not depend on the choice of admissible reading.

A Young diagram Y is called an (m, n) -hook Young diagram if the number of boxes in $(m+1)$ -th row is at most n ; i.e., there is no box in the $(m+1, n+1)$ -th position. We denote by $H(m, n)$ the set of all (m, n) -hook Young diagrams. Note that a Young diagram Y can be made into a $\mathfrak{gl}(m, n)$ -semistandard tableau if and only if Y is an (m, n) -hook Young diagram. To describe the decomposition of the tensor product of $U_q(\mathfrak{gl}(m, n))$ -crystals, we need the following definitions.

Let Y be a skew Young diagram with an admissible order A and let Q be a semistandard skew tableau of shape Y with entries in \mathbb{N} . The *word of Q with respect to A* is defined to be

$$w_A(Q) = (i_1, i_2, \dots, i_N),$$

where $R_A(Q) = \boxed{i_1} \otimes \dots \otimes \boxed{i_N}$, and the *content of Q* is defined to be

$$\text{cont}(Q) = (\mu_i)_{i \in \mathbb{N}},$$

where μ_i is the number of i 's in Q . A finite sequence (i_1, \dots, i_N) is called a *lattice permutation* if for every $i \in \mathbb{N}$ and k with $1 \leq k \leq N$, the number of occurrences of i in (i_1, \dots, i_k) is greater than or equal to the number of occurrences of $i+1$ in (i_1, \dots, i_k) .

Definition 3.2. Let Y, W, Z be (m, n) -hook Young diagrams with $|Y| + |W| = |Z|$ and let A be an admissible order on Z/Y . A $U_q(\mathfrak{gl}(m, n))$ -Littlewood-Richardson tableau associated with (Y, W, Z) and A is a semistandard skew tableau Q with entries in \mathbb{N} satisfying the following conditions:

- (i) $\text{sh}(Q) = Z/Y$,
- (ii) $\text{cont}(Q) = W$,
- (iii) $w_A(Q)$ is a lattice permutation.

We denote by $LR(Y, W)^Z[A]$ the set of all $U_q(\mathfrak{gl}(m, n))$ -Littlewood-Richardson tableaux associated with (Y, W, Z) and A . Using the insertion scheme for $U_q(\mathfrak{gl}(m, n))$ -crystals, it was shown in [6, Proposition 4.13] that $LR(Y, W)^Z[A]$ are the same for all admissible orders on Z/Y . The number $N_{Y, W}^Z = |LR(Y, W)^Z[A]|$ is called the $U_q(\mathfrak{gl}(m, n))$ -Littlewood-Richardson coefficient associated with (Y, W, Z) . Moreover, [6, Theorem 4.16] yields the decomposition of the tensor product of $U_q(\mathfrak{gl}(m, n))$ -crystals:

$$(3.1) \quad \mathfrak{B}(Y) \otimes \mathfrak{B}(W) \cong \bigoplus_{Z \in H(m, n)} \mathfrak{B}(Z)^{\oplus N_{Y, W}^Z}.$$

We now state and prove our main theorem.

Theorem 3.3. *Let Y, W, Z be (m, n) -hook Young diagrams with $|Y| + |W| = |Z|$, and let A, A' be admissible orders on $W, Z/Y$, respectively. Then there exists a natural bijection*

$$\tilde{\Phi} : \mathbf{P}(Z/Y, W; A, A') \longrightarrow LR(Y, W)^Z[A']$$

defined by

$$(3.2) \quad \tilde{\Phi}(f)_{ij} = f_1(i, j) \quad \text{for } f = (f_1, f_2) \in \mathbf{P}(Z/Y, W; A, A').$$

The inverse map

$$\tilde{\Psi} : LR(Y, W)^Z[A'] \longrightarrow \mathbf{P}(Z/Y, W; A, A')$$

is given by

$$(3.3) \quad \tilde{\Psi}(Q)(i, j) = (Q_{ij}, p(Q; i, j)) \quad \text{for } Q \in LR(Y, W)^Z[A'].$$

Proof. Write $W = (W_1 \geq W_2 \geq \cdots \geq W_k > 0)$ and $Z = (Z_1 \geq Z_2 \geq \cdots \geq Z_h > 0)$. Set $r = \max(k, h)$ and consider the set $\mathbf{B}(Z/Y)$ of semistandard skew tableaux of shape Z/Y with entries in $\{1, 2, \dots, r\}$. We claim

$$LR(Y, W)^Z[A'] = \mathbf{B}(Z/Y)_\emptyset^W[A'].$$

Let $Q \in LR(Y, W)^Z[A']$. Then Q is a semistandard skew tableau of shape Z/Y with entries in \mathbb{N} such that $\text{cont}(Q) = W$ and $w_{A'}(Q)$ is a lattice permutation. Write $R_{A'}(Q) = \boxed{i_1} \otimes \cdots \otimes \boxed{i_N}$. Since $w_{A'}(Q) = (i_1, \dots, i_N)$ is a lattice permutation, $\emptyset[i_1, \dots, i_r]$ is a Young diagram for all $r = 1, \dots, N$. Moreover, since $\text{cont}(Q) = W$, we have $\emptyset[i_1, \dots, i_N] = W$. Hence $Q \in \mathbf{B}(Z/Y)_\emptyset^W[A']$.

Conversely, if $T \in \mathbf{B}(Z/Y)_\emptyset^W[A']$, then a similar argument shows $T \in LR(Y, W)^Z[A']$, which proves our claim.

By Theorem 2.7, there exist natural bijections

$$\Phi : \mathbf{P}(Z/Y, W; A, A') \longrightarrow \mathbf{B}(Z/Y)_\emptyset^W[A'],$$

$$\Psi : \mathbf{B}(Z/Y)_\emptyset^W[A'] \longrightarrow \mathbf{P}(Z/Y, W; A, A')$$

defined by

$$\Phi(f)_{ij} = f_1(i, j) \quad \text{for } f = (f_1, f_2) \in \mathbf{P}(Z/Y, W; A, A'),$$

$$\Psi(Q)(i, j) = (Q_{ij}, p(Q; i, j)) \quad \text{for } Q \in \mathbf{B}(Z/Y)_\emptyset^W[A'].$$

Since $LR(Y, W)^Z[A'] = \mathbf{B}(Z/Y)_\emptyset^W[A']$, we are done. \square

Corollary 3.4. *Let Y, W, Z be (m, n) -hook Young diagrams and let A, A' be admissible orders on $W, Z/Y$, respectively. Write $W = (W_1 \geq W_2 \geq \cdots \geq W_k > 0)$, $Z = (Z_1 \geq Z_2 \geq \cdots \geq Z_h > 0)$ and set $r = \max(k, h)$. Then there exists a natural bijection*

$$\hat{\Phi} : LR(Y, W)^Z[A'] \longrightarrow \mathbf{B}(W)_Y^Z[A]$$

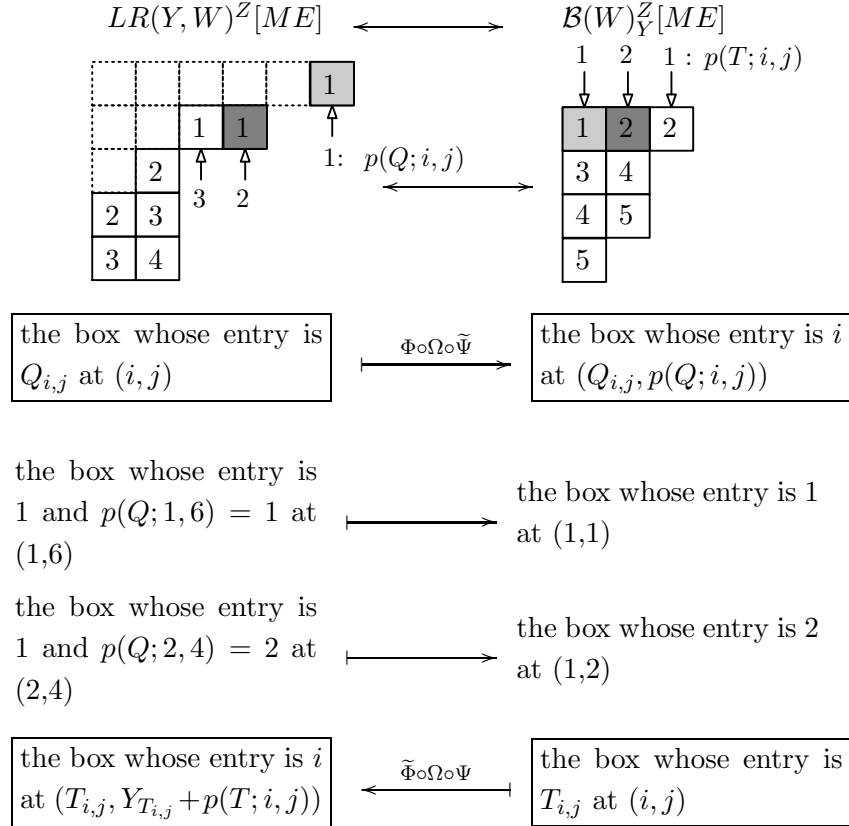
between the set of $U_q(\mathfrak{gl}(m, n))$ -Littlewood-Richardson tableaux and the set of $U_q(\mathfrak{gl}(r))$ -Littlewood-Richardson tableaux. In particular, we have $N_{Y,W}^Z = c_{Y,W}^Z$.

Proof. By the definition of admissible pictures, the map

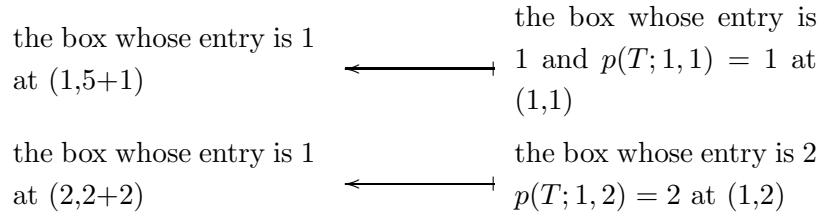
$$\begin{array}{ccc} \Omega : \mathbf{P}(Z/Y, W; A, A') & \longrightarrow & \mathbf{P}(W, Z/Y; A', A) \\ f & \longmapsto & f^{-1} \end{array}$$

is a bijection. Hence the composition $\widehat{\Phi} = \Phi \circ \Omega \circ \widetilde{\Psi}$ is the desired bijection. \square

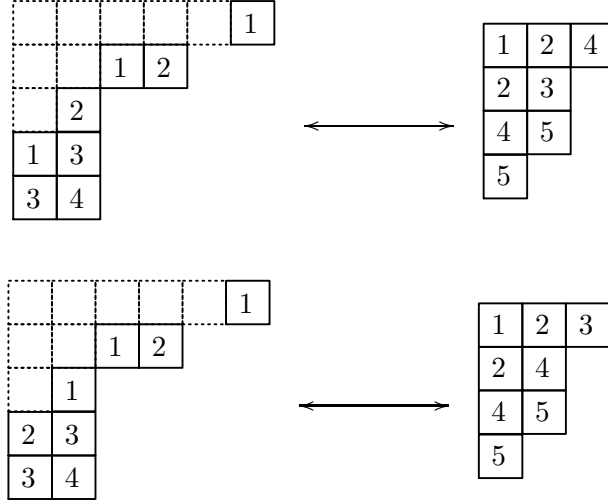
Example 3.5. Let $Y = (5, 2, 1)$, $W = (3, 2, 2, 1)$ and $Z = (6, 4, 2, 2, 2)$ be $(3, 3)$ -hook Young diagrams. We present the 1-1 correspondence between $LR(Y, W)^Z[ME]$ for $\mathfrak{gl}(3, 3)$ and $\mathbf{B}(W)_Y^Z[ME]$ for $\mathfrak{gl}(5)$ given in the proof of Corollary 3.4.



For instance,



Similarly, we have the following correspondence:



Remark. The correspondence in Corollary 3.4 is the same as that of Theorem C in the appendix of [7].

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